1 Preliminaries

1.1 Hyper Sphere and Hyper Ball

In this section we introduce notation for the fundamental geometrical objects that we will use thorough this proof. We will freely make use of terms like *volume*, *area* or *surface* to guide the reader along our line of though. Nonetheless, these concepts are misleading and somewhat overlapping, especially when discussing high-dimensional geometry. Thus, we will also give explicit definition for these terms in order to avoid any misunderstanding of our argument.

Definition 1 (n-dimensional Sphere)**.** *We call n-sphere of center $O \in \mathbb{R}^n$ and radius $R \in \mathbb{R}$ and write $\mathcal{S}(O, R) \subset \mathbb{R}^n$ the subset*

$$\mathcal{S}(O, R) \doteq \{x \in \mathbb{R}^n : \|x - O\|_2 = R\}$$

Definition 2 (n-dimensional Ball)**.** *We call n-ball of center $O \in \mathbb{R}^n$ and radius $R \in \mathbb{R}$ and write $\mathcal{B}(O, R) \subset \mathbb{R}^n$ the subset*

$$\mathcal{B}(O, R) \doteq \{x \in \mathbb{R}^n : \|x - O\|_2 < R\}$$

Definition 3 (Surface of a spehe)**.** *We call* Surface *of the n-sphere $\mathcal{S}(O, R)$ and write $V(\mathcal{S}(O, R))$ the $n - 1$ dimensional volume*

$$V(\mathcal{S}(O, R)) \doteq \Pi_n^s \times R^{n-1}$$

Where Π_n^s is a constant factor depending only on n

Definition 4 (Volume of a Ball)**.** *We call* Volume *of the n-ball $\mathcal{B}(O, R)$ and write $V(\mathcal{B}(O, R))$ the n-dimensional volume*

$$V(\mathcal{B}(O, R)) \doteq \int_0^R V(\mathcal{S}(O, r))dr$$

That is

$$\begin{aligned} V(\mathcal{B}(O, R)) &= \int_0^R \Pi_n^s R^{n-1} dr \\ &= \frac{\Pi_n^s R^n}{n} \\ &= \Pi_n R^n \end{aligned}$$

Where $\Pi_n \doteq \frac{\Pi_n^s}{n}$ is a constant factor depending only on n.

1.2 Hyper-Cone

From these core definitions, we can now introduce (Hyper)-cone and some of their core properties. Intuitively, an Hyper-cone of dimension $n + 1$, center O, radius R and height H is a sequence of n-Ball of linearly decreasing radius between R and 0, each one living on a difference "slice" of $\mathbb{R}^{n+1}$ between 0 and H.

Definition 5 (Hyper-cone)**.** *We call Hyper-cone of dimension* $n+1$*, base* $\mathcal{B}(O,R) \subset \mathbb{R}^n$ *and height* H *the set :*

$$H \doteq \bigcup_{\forall h \in [0,H]} \mathcal{B}\left(O + (H-h)\mathbf{u}_{n+1}, \frac{h}{H}R\right)$$

Where $\mathbf{u}_{n+1}$ *is the* $n+1^{th}$ *base vector of* $\mathbb{R}^{n+1}$*.*

Alternatively, we can define the *apex* $Z \doteq O + H \times \mathbf{u}_{n+1}$ of the hyper-cone and give the following definition :

Definition 6 (Hyper-cone (2))**.** *We call Hyper-cone of dimension* $n+1$*, base* $\mathcal{B}(O,R) \subset \mathbb{R}^n$ *and apex* Z *the convex hull* $\textbf{\textit{conv}}\left(\{\mathcal{B}(O,R), z\}\right)$*.*

We are now ready to state the core properties of Hyper-cone that we will use in the remaining of the proof.

Let's start with the volume of a Hyper-cone

Definition 7 (Volume of Hyper-cone)**.** *Given an Hyper-cone* $C \in \mathbb{R}^{n+1}$ *of dimension* $n+1$*, base* $\mathcal{B}(O,R) \subset \mathbb{R}^n$ *and height* H *we call* volume *and write* $V(C)$ *the* $n+1$*-dimensional volume :*

$$V(C) \doteq \int_0^H V\left(\mathcal{B}\left(O + (H-h)\mathbf{u}_{n+1}, \frac{h}{H}R\right)\right) dh$$

Proposition 1. *The volume of the Hyper-cone* $C \subset \mathbb{R}^{n+1}$ *of dimension* $n+1$*, base* $\mathcal{B}(O,R) \subset \mathbb{R}^n$ *and height* H *is*

$$V(C) = \frac{\Pi_n R^n}{n+1} H$$

Proof. From the definition of volume of a sphere we have $\mathrm{V}(\mathcal{B}(O,R)) \doteq \Pi_n R^n$. Substituting R by $h/_H R$ and from the definition of the volume of a Hyper-cone we have

$$\begin{aligned}
\mathrm{V}(C) &= \int_0^H \Pi_n \left(\frac{h}{H}R\right)^n dh \\
&= \frac{\Pi_n R^n}{H^n} \int_0^H h^n dh \\
&= \frac{\Pi_n R^n}{H^n} \times \frac{H^{n+1}}{n+1} \\
&= \frac{\Pi_n R^n}{n+1} H
\end{aligned}$$

□

1.3 Center of Mass

With the previous definition formally stated, we can now go one step further and define the *center of mass* of *center of gravity*

Definition 8 (Center of gravity)**.** *For a given set $X \subset \mathbb{R}^n$ we call* center of gravity [1] *and write $\boldsymbol{cg}(X) \in \mathbb{R}^n$ the point :*

$$\boldsymbol{cg}(X) = \int_{x \in X} x dx$$

Proposition 2. *Let S a n-dimensional sphere such that $S \doteq \mathcal{S}(O, R)$. Then $\boldsymbol{cg}(S) = O$.*

Proof. Without loss of generality, assume that $O = 0$. Then, $S = \{x \in \mathbb{R}^n : \|x\|_2 = R\}$. Since $\|x\|_2 = \|-x\|_2$ it is clear that $\forall x \in \mathbb{R}^n : x \in S \Leftrightarrow -x \in S$. Thus, we can rewrite $\mathbf{cg}(S)$ as

$$\mathbf{cg}(S) = \frac{1}{\mathrm{V}(S)} \int_{x \in S} -x dx$$

Thus

$$\begin{aligned} 2\mathbf{cg}(S) &= \frac{1}{\mathrm{V}(S)} \left(\int_{x \in S} x dx + \int_{x \in S} -x dx \right) \\ &= \frac{1}{\mathrm{V}(S)} \int_{x \in S} x - x dx \\ &= 0 \end{aligned}$$

□

Proposition 3. *Let B a n-dimensional ball such that $B \doteq \mathcal{B}(O, R)$. Then $\boldsymbol{cg}(B) = O$.*

Proof. Remind that B can be seen as a collection of concentric n-sphere of center O and radii between 0 and R. Notably, this construction is implicit in the definition of $\mathrm{V}(B)$. Then, we can rewrite $\mathbf{cg}(B)$ as

$$\begin{aligned} \mathbf{cg}(B) &= \frac{1}{\mathrm{V}(B)} \int_0^R \mathbf{cg}(\mathcal{S}(O, r)) \mathrm{V}(\mathcal{S}(O, r)) dr \\ &= O \end{aligned}$$

Where the last line come from Proposition 2. □

Proposition 4 (Center of Gravity of an Hyper-cone)**.** *Let C a $n+1$ dimensional Hyper-cone ($C \subset \mathbb{R}^{n+1}$) of base $\mathcal{B}(O, R) \subset \mathbb{R}^n$ and apex Z such that $\|Z - O\|_2 = H$. Then, $\boldsymbol{cg}(C)$ is located on the segment $[O; Z]$ at a distance $H/_{n+2}$ of O.*

[1] For a more complete definition, we should take into account the mass distribution over X and the density of X. Although, in an effort to keep things simple, we assume a density of 1 and a uniformly distributed mass.

Proof. Without loss of generality, we assume that $O = 0$ and $Z = H\mathbf{u}_{n+1}$. Where $\mathbf{u}_{n+1}$ is the $n+1^{\text{th}}$ base vector of $\mathbb{R}^{n+1}$. By definition, C is a collection of ball, and in a similar way as before we can rewrite $\mathbf{cg}(C)$ as :

$$\mathbf{cg}(C) = \frac{1}{\mathrm{V}(C)} \int_0^H \mathbf{cg}(\mathcal{B}((H-h)\mathbf{u}_{n+1}, h/_H R))\mathrm{V}(\mathcal{B}((H-h)\mathbf{u}_{n+1}, h/_H R))dh$$

From this and Proposition3 it is clear that $\mathbf{cg}(C)$ lies on the segment $[O; Z]$. The remaining of the proof came by explicitly calculating $\mathbf{cg}(C)$.

$$\begin{aligned}
\mathbf{cg}(C) &= \frac{1}{\mathrm{V}(C)} \int_0^H (H-h)\mathbf{u}_{n+1} \times \mathrm{V}(\mathcal{B}((H-h)\mathbf{u}_{n+1}, h/_H R))dh \qquad \text{(Prop. 3)} \\
&= \frac{1}{\mathrm{V}(C)} \int_0^H (H-h)\mathbf{u}_{n+1} \frac{\Pi_n R^n}{H^n} h^n dh \qquad \text{(Volume of a Ball)} \\
&= \frac{1}{\mathrm{V}(C)} \left[\int_0^H \frac{\Pi_n R^n H \mathbf{u}_{n+1}}{H^n} h^n dh - \int_0^H \frac{\Pi_n R^n \mathbf{u}_{n+1}}{H^n} h^{n+1} dh \right] \\
&= \frac{1}{\mathrm{V}(C)} \left[\frac{\Pi_n R^n \mathbf{u}_{n+1}}{H^{n-1}} \int_0^H h^n dh - \frac{\Pi_n R^n \mathbf{u}_{n+1}}{H^n} \int_0^H h^{n+1} dh \right] \\
&= \frac{1}{\mathrm{V}(C)} \left[\frac{\Pi_n R^n H^{n+1} \mathbf{u}_{n+1}}{H^{n-1}(n+1)} - \frac{\Pi_n R^n H^{n+2} \mathbf{u}_{n+1}}{H^n(n+2)} \right] \\
&= \frac{1}{\mathrm{V}(C)} \left[\frac{\Pi_n R^n H^2 \mathbf{u}_{n+1}}{n+1} - \frac{\Pi_n R^n H^2 \mathbf{u}_{n+1}}{n+2} \right] \\
&= \frac{n+1}{\Pi_n R^n H} \left[\frac{\Pi_n R^n H^2 \mathbf{u}_{n+1}}{n+1} - \frac{\Pi_n R^n H^2 \mathbf{u}_{n+1}}{n+2} \right] \\
&\qquad \text{(Volume of a Hyper-cone)} \\
&= \left(H - H\frac{n+1}{n+2} \right) \mathbf{u}_{n+1} \\
&= \left(1 - \frac{n+1}{n+2} \right) H\mathbf{u}_{n+1} \\
&= \left(\frac{n+2-n-1}{n+2} H\mathbf{u}_{n+1} \right) \\
&= \left(\frac{H}{n+2} \right) \mathbf{u}_{n+1}
\end{aligned}$$

That is to say, $\mathbf{cg}(C)$ is on the segment $[O; Z]$ at a distance $H/_{n+2}$ of O. □

1.4 Hyperplane and Halfspace

Definition 9. *We call* n-Hyperplane *of normal* $w \in \mathbb{R}^n$ *and offset* $b \in \mathbb{R}$ *and write* $W(w, b) \subset \mathbb{R}^n$ *the subset :*

$$W(w, b) \doteq \{x \in \mathbb{R}^n : \langle w, x \rangle + b = 0\}$$

Definition 10. *We call* Positive n-Halfspace *of n-Hyperplane $W(w,b) \subset \mathbb{R}^n$ and write $W^+(w,b) \subset \mathbb{R}^n$ the subset*

$$W^+(w,b) \doteq \{x \in \mathbb{R}^n : \langle w, x \rangle + b \geq 0\}$$

Definition 11. *We call* Negative n-Halfspace *of n-Hyperplane $W(w,b) \subset \mathbb{R}^n$ and write $W^-(w,b) \subset \mathbb{R}^n$ the subset*

$$W^-(w,b) \doteq \{x \in \mathbb{R}^n : \langle w, x \rangle + b \leq 0\}$$

Additionally, note that $W(w,b) \subset W^+(w,b)$ but $W(w,b) \not\subset W^-(w,b)$.

Definition 12. *For any subset $X \subset \mathbb{R}^n$ and any n-Hyperplane $W \subset \mathbb{R}^n$ we call* Positive Partition *and write $X^+ \subset \mathbb{R}^n$ the subset*

$$X^+ \doteq X \cap W^+$$

Definition 13. *For any subset $X \subset \mathbb{R}^n$ and any n-Hyperplane $W \subset \mathbb{R}^n$ we call* Negative Partition *and write $X^+ \subset \mathbb{R}^n$ the subset*

$$X^- \doteq X \cap W^-$$

Proposition 5 (Volume reduction of Hyper-Cone)**.** *For any $(n+1)$-Hyper-cone of base $\mathcal{B}(O,R)$, apex Z and Height H, let set $W_{\boldsymbol{cg}(C)} \doteq W(\mathbf{u}_{n+1}, H/_{n+2})$ the $(n+1)$-Hyperplane passing by $\boldsymbol{cg}(C)$ (i.e. $\boldsymbol{cg}(C) \in W_{\boldsymbol{cg}(C)}$) and parallel to $\mathbb{R}^n$. Then,*

$$V(C^+) = V(C) \left[\frac{1}{\left(1 + \frac{1}{n+1}\right)^{n+1}} \right] \geq V(C)e^{-1}$$

Proof. We start by proving the right-hand side of the relation. Let set $n' = n+1$ and divide both side by V(C) then we can rewrite it as

$$\frac{1}{(1 + \frac{1}{n'})^{n'}} \geq e^{-1}$$

From the usual definition of e we have that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n'}\right)^{n'} = e$$
$$\Leftrightarrow \lim_{n' \to \infty} \frac{1}{(1 + \frac{1}{n'})^{n'}} = e^{-1}$$

And by simple induction argument we can show that

$$\frac{1}{(1 + \frac{1}{n'})^{n'}} \geq \frac{1}{\left(1 + \frac{1}{n'+1}\right)^{n'+1}}$$

Therefore

$$\frac{1}{\left(1+\frac{1}{n'}\right)^{n'}} \geq \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n'}\right)^{n'}} = e^{-1}$$

Finally, the left-hand side of relation is obtained by direct calculation of V(C+). The general idea is the same than the calculation of V(C) but, instead of integrating over the entire height we stop at $H(1-1/_{n+2})$, thus ignoring C^-. Besides, without loss of generality, we assume that $O = 0$ and that $Z = H\mathbf{u}_{n+1}$.

$$\begin{aligned}
\mathrm{V}(C^+) &= \int_0^{H\left(1-\frac{1}{n+2}\right)} \mathrm{V}\left[\mathcal{B}\left((H-h)\mathbf{u}_{n+1}, \frac{R}{H}h\right)\right] dh \\
&= \int_0^{H\left(1-\frac{1}{n+2}\right)} \Pi_n \frac{R^n}{H^n} h^n dh \\
&= \frac{\Pi_n R^n}{H^n} \int_0^{H\left(1-\frac{1}{n+2}\right)} h^n dh \\
&= \frac{\Pi_n R^n}{H^n} \times \frac{H^{n+1}}{n+1} \times \left(1 - \frac{1}{n+2}\right)^{n+1} \\
&= \frac{\Pi_n R^n H}{n+1} \times \left(1 - \frac{1}{n+2}\right)^{n+1} \\
&= \mathrm{V}(C)\left(1 - \frac{1}{n+2}\right)^{n+1} \\
&= \mathrm{V}(C)\left(\frac{n+1}{n+2}\right)^{n+1} \\
&= \mathrm{V}(C)\left(\frac{(n+1)\times\frac{1}{n+1}}{(n+2)\times\frac{1}{n+1}}\right)^{n+1} \\
&= \mathrm{V}(C)\left(\frac{1}{\frac{n+2}{n+1}}\right)^{n+1} \\
&= \mathrm{V}(C)\left(\frac{1}{1+\frac{1}{n+1}}\right)^{n+1} \\
&= \mathrm{V}(C)\left[\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}\right]
\end{aligned}$$

□

2 State of the Art

This section is basically a rewriting of the proof of (Grunbaum, 1960) in a more comprehensive way.

2.1 Setting

Let K a (full dimensional) convex set in $\mathbb{R}^{n+1}$.

Definition 14. *For any convex body $K \in \mathbb{R}^{n+1}$ we say that K is* Spherically Symmetric along the unit vector $\mathbf{u}$ *if and only if $\forall \lambda \in \mathbb{R}$ the cut of K by the hyperplane $W(\mathbf{u}, \lambda)$ (i.e. $K \cap W(\mathbf{u}, \lambda)$) is a n dimensional hypersphere of center $\lambda \mathbf{u}$*

2.2 Theorem

We want to prove the following theorem :

Theorem 1. *For any convex set $K \subset \mathbb{R}^{n+1}$, and any hyperplane W. If $\mathbf{cg}(K) \in K^+$ then*

$$V(K^+) \geq e^{-1} \times V(K)$$

Proof.

Note 1 (Points along $\mathbf{u}_{n+1}$)**.** *This proof will revolve around key points located on the $n+1^{th}$ axis of $\mathbb{R}^{n+1}$ of base vector $\mathbf{u}_{n+1}$. In an attempt to avoid over-burdening the notation, we will treat these points as number along the real line when context is clear. Therefore, if $x = \lambda_1 \mathbf{u}_{n+1}$ and $y = \lambda_2 \mathbf{u}_{n+1}$ we will freely write $x > y$ if $\lambda_1 > \lambda_2$.*

Let W the hyperplane such that $W = \arg\min_W \mathrm{V}(K^+)$ such that $\mathbf{cg}(K) \in K^+$. It is easy to see that $\mathbf{cg}(K) \in W$: if $\mathbf{cg}(K) \notin W$ you can always reduce $\mathrm{V}(K^+)$ by shifting W toward $\mathbf{cg}(K)$. Without loss of generality, let's say that $\mathbf{cg}(K) = 0$ the origin of $\mathbb{R}^{n+1}$ and that $\mathbf{u}_{n+1}$, the $n+1^{\text{th}}$ dimensional base vector of $\mathbb{R}^{n+1}$, is the normal vector of W with $b \in \mathbb{R}_-$ set accordingly such that $W = W(\mathbf{u}_{n+1}, b)$.

In order to ease the comprehension of the proof, we make the following assumption that we will lift later on.

Assumption 1. *K is a convex body which is Spherically Symmetric along $\mathbf{u}_{n+1}$*

A direct implication of this is that $\mathbf{cg}(K) = \mathbf{cg}(K \cap W)$. In other words, $\mathbf{cg}(K)$ is the center of the $n-1$ dimensional sphere $K \cap W$ (see, for example, the argument of Prop. 4).

Let C^+ the cone of base $K \cap W$ and apex Z such that $C^+ \subset W^+$ and $\mathrm{V}(C^+) = \mathrm{V}(K^+)$. By construction we have that $Z = H\mathbf{u}_{n+1}$ where H is the height of C^+.

Moreover either :

- $K^+ = C^+$ and Z is the apex of K^+
- $Z \notin K^+$

To prove that, remember that each *slice* $K \cap W(\mathbf{u}_{n+1}, \lambda)$ of K along the $n+1$ axis is a sphere. We look at the function $r()$ which maps each value of λ with the radius of the corresponding *slice*. If K^+ is a Hyper-cone, by definition $r()$ is linear. If $r()$ has any convex part, then there exists an arc $[r(\lambda_1), r(\lambda_2)]$ which is not in K^+ and therefore K^+ is not a convex set. Therefore $r()$ is strictly concave, then, by definition of concave function, either $C^+ \subset K^+$ and therefore $\mathrm{V}(K^+) \geq \mathrm{V}(C^+)$ –Which is a contradiction– or $Z \notin K^+$.

The (potentially) more elongated shape of C^+ means that $\mathbf{cg}(C^+)$ is on the closed segment $[\mathbf{cg}(K^+), Z]$.

In other word, and according to Note 1 :

$$0 = \mathbf{cg}(K) \leq \mathbf{cg}(K^+) \leq \mathbf{cg}(C^+) \leq Z$$

We now define C^- by extending C^+ such that $C \doteq C^- \cup C^+$ is a cone of apex Z and $\mathrm{V}(C^-) = \mathrm{V}(K^-)$. Therefore,

$$\begin{aligned} \mathrm{V}(C) &= \mathrm{V}(C^+) + \mathrm{V}(C^-) \\ &= \mathrm{V}(K^+) + \mathrm{V}(K^-) \\ &= \mathrm{V}(K) \end{aligned}$$

Once again, we are interested in the relative position of $\mathbf{cg}(K^-)$ and $\mathbf{cg}(C^-)$. We invoke the same arguments than before and claim that, in a similar way :

$$\mathbf{cg}(K^-) \leq \mathbf{cg}(C^-) \leq 0 = \mathbf{cg}(C)$$

Remark 1. *The proof for this is a little more tricky this time though. Part of this is due to the lack of common base for K^- and C^- and the fact that C^- is not a Hyper-cone in itself. A possible start is to consider the radius increase along the reverse axis $\overline{\mathbf{u}_{n+1}} \doteq -\mathbf{u}_{n+1}$ and replicate the previous argument.*

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \doteq {}^{\mathrm{V}(K^+)}/_{\mathrm{V}(K)}$ and $\beta \doteq \mathrm{V}(K^-)/_{\mathrm{V}(K)}$. Then

$$\mathrm{V}(K) = \alpha \mathrm{V}(K^+) + \beta \mathrm{V}(K^-)$$

Or, alternatively

$$\mathrm{V}(C) = \alpha \mathrm{V}(C^+) + \beta \mathrm{V}(C^-)$$

Combining the previous inequalities, we have that

$$\mathbf{cg}(K) \leq \mathbf{cg}(C)$$

Moreover, we know from Proposition 4 that $\mathbf{cg}(C)$ is at a distance $H/_{n+1+2}$ of its base.

Let $\widetilde{W} \doteq W(\mathbf{u}_{n+1}, \tilde{b})$ such that $\mathbf{cg}(C) \in \widetilde{W}$ and write $\widetilde{C^+}$ the positive partition of C by $\widetilde{W}$, that is $\widetilde{C^+} \doteq \widetilde{W^+} \cap C$.

From Proposition 5 we have that $\mathrm{V}\left(\widetilde{C^+}\right) \geq e^{-1}\mathrm{V}(C)$. Moreover, because of $\mathbf{cg}(C) \geq \mathbf{cg}(K)$ we have that $\mathrm{V}(C^+) \geq \mathrm{V}\left(\widetilde{C^+}\right)$.

Putting all of this together we get that

$$\begin{aligned}
\mathrm{V}(K^+) &= \mathrm{V}(C^+) \\
&\geq \mathrm{V}\left(\widetilde{C^+}\right) \\
&\geq e^{-1}\mathrm{V}(C) \\
&= e^{-1}\mathrm{V}(K)
\end{aligned}$$

Finally, all we have left is to deal with Assumption 1. This is simply tackled by remarking that, by definition, spherical symmetrization preserve volumes along its axis. Thus, for any K of any convex shape it suffices to consider the spherical symmetrization of K : $\mathbf{sym}_S(K)$ and we have

$$\mathrm{V}(K^+) = \mathrm{V}(\mathbf{sym}_S(K^+)) \geq e^{-1}\mathrm{V}(\mathbf{sym}_S(K)) = \mathrm{V}(K)$$

□

3 Generalized Volume Reduction

This section prove the generalization of the previous result.

Theorem 2. *For any $d \in \mathbb{N}$, any convex set $K \subset \mathbb{R}^{n+1}$, any hyperplane W of normal vector $\mathbf{a}$ and any $\overline{X} \doteq \boldsymbol{cg}(K) + \mathbf{a}\lambda \frac{(n+1)\,V(K)\left[1-\frac{1}{n+1}\right]}{\Pi_n R_K^n}$. Where R_K is the radius of the of $\boldsymbol{sym}_S(K) \cap W$ If $\overline{X} \in K^+$ then*

$$V(K^+) \geq (1-\lambda)^{n+1} \times V(K)$$

Proof. The proof essentially follows the same line of thoughts than the one of Grunbaum and the construction is similar.

Namely, we construct C, C^+ and C^- with respect to W as in the previous proof.

Once again, we will compare points that are along the axis of $\mathbf{a}$ and we will consider them as real for readability concerns.

We are interested in the distance between the apex Z of C and $\overline{X}$ as this distance is characteristic of the volume of the volume of C^+ :

$$
\begin{aligned}
Z - \overline{X} &= Z - \mathbf{cg}(K) - \lambda \frac{(n+1)\mathrm{V}(K)\left[1 - \frac{1}{n+1}\right]}{\Pi_n R_K^n} \\
&\geq \mathbf{cg}(C) - \lambda \frac{(n+1)\mathrm{V}(K)\left[1 - \frac{1}{n+1}\right]}{\Pi_n R_K^n} \\
&\geq \mathbf{cg}(C) - \lambda \frac{(n+1)\mathrm{V}(K)\left[1 - \frac{1}{n+1}\right]}{\Pi_n R^n} \\
&= \left[1 - \frac{1}{n+1}\right] H - \lambda \frac{(n+1)\mathrm{V}(K)\left[1 - \frac{1}{n+1}\right]}{\Pi_n R^n} \\
&= \left[1 - \frac{1}{n+1}\right] H - \lambda \left[1 - \frac{1}{n+1}\right] H \\
&= (1-\lambda) H \left[1 - \frac{1}{n+1}\right]
\end{aligned}
$$

Define $\overline{X'}$ such that $Z - \overline{X'} \doteq (1-\lambda)H\left[1 - \frac{1}{n+1}\right]$. On a side note, remark that

$$
\overline{X'} = \mathbf{cg}(C) - \lambda \frac{(n+1)\mathrm{V}(K)\left[1 - \frac{1}{n+1}\right]}{\Pi_n R_K^n}
$$

Similarly to C^+, define C'^+ with respect to $\overline{X'}$. The previous inequality implies that $C'^+ \subseteq C^+$, thus

$$
\mathrm{V}(K^+) = \mathrm{V}(C^+) \geq \mathrm{V}(C'^+)
$$

And the theorem comes from the closed form of $\mathrm{V}(C'^+)$

$$
\begin{aligned}
\mathrm{V}(C'^+) &= \int_0^{\overline{X'}} \left(\mathcal{B}\left(O + (H-h)\mathbf{u}_{n+1}, \frac{h}{H} R \right) \right) dh \\
&= (1-\lambda)^{n+1} \left[1 - \frac{1}{n+1}\right]^{n+1} \mathrm{V}(C) \\
&\geq (1-\lambda)^{n+1} e^{-1} \mathrm{V}(C)
\end{aligned}
$$

□